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# Matrix realizations of exceptional superconformal algebras

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**Abstract**

We give a general construction of realizations of the contact superconformal algebras $K(2)$ and $\hat{K}'(4)$, and the exceptional superconformal algebra $CK_6$ as subsuperalgebras of matrices over a Weyl algebra of size $2^N \times 2^N$, where $N = 1, 2$ and 3. We show that there is no such realization for $K(2N)$, if $N \geq 4$.


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## 1. Introduction

Superconformal algebras are superextensions of the Virasoro algebra. They play an important rôle in string theory, conformal field theory and mirror symmetry, and have been extensively studied by mathematicians and physicists. A *superconformal algebra* is a simple complex Lie superalgebra, spanned by the coefficients of a finite family of pairwise local fields $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, one of which is the Virasoro field $L(z)$ [3,8,9]. It can also be described in terms of vector fields and symbols of differential operators.

An important class of superconformal algebras are the Lie superalgebras $K(N)$ of contact vector fields on the supercircle $S^{1|N}$ with even coordinate $t$ and $N$ odd coordinates. The superalgebra $K(N)$ is characterized by its action on a contact 1-form [3,4,9,13]. It is spanned by $2^N$ fields. These superalgebras are also known to physicists as $SO(N)$ superconformal algebras [1,2]. They are especially interesting when $N$ is small. The universal central extension of $K(2)$ is isomorphic to the "$N = 2$ superconformal algebra". The superalgebra $K'(4)$ has three independent central extensions, one of which is given by the Virasoro 2-cocycle and is isomorphic to the "big $N = 4$ superconformal algebra", see [1,2]. In this work we consider a different non-trivial central extension $\hat{K}'(4)$ of $K'(4)$. Note that $K(N)$ has no non-trivial central extensions if $N > 4$ [13]. The superalgebra $K(6)$ contains the exceptional "$N = 6$ superconformal algebra" as a subsuperalgebra. It constitutes "one half" of $K(6)$, and it is also denoted by $CK_6$, see [3,6,9–12,22–24].

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In [17,18], we proved that for every $N \geq 0$, there exists an embedding of $K(2N)$ into the Poisson superalgebra $P(2N)$ of pseudodifferential symbols on the supercircle $S^{1|N}$. $P(2N) = P \otimes \Lambda(2N)$, where $P$ is the Poisson algebra of functions on the cylinder $T^*S^1 \setminus S^1$, and $\Lambda(2N)$ is the Grassmann algebra.

It is a remarkable fact that $K(2)$, $\hat{K}'(4)$ and $CK_6$, for $N = 1, 2$ and 3, respectively, admit embeddings into the family $P_h(2N)$ of Lie superalgebras of pseudodifferential symbols on $S^{1|N}$, which contracts to $P(2N)$ [17,18]. Such embeddings allow us to obtain realizations of these superconformal algebras as subsuperalgebras of matrices of size $2 \times 2$, $4 \times 4$, and $8 \times 8$, respectively, over a Weyl algebra $\mathcal{W} = \sum_{i \geq 0} \mathcal{A} d^i$, where $\mathcal{A} = \mathbb{C}[t, t^{-1}]$ and $d = \frac{\partial}{\partial t}$, see [19,20].

In [15,16] Martinez and Zelmanov obtained $CK_6$ as a particular case of superalgebras $CK(R, d)$, where $R$ is an associative commutative superalgebra with an even derivation $d$. They also realized $CK_6$ as a subsuperalgebra of matrices of size $8 \times 8$ over $\mathcal{W}$.

In this work, we give a general construction of matrix realizations of $K(2)$, $\hat{K}'(4)$ and $CK_6$. Note that a semi-direct sum of the Lie algebra $\mathfrak{o}(2N, \mathbb{C})$ and the Heisenberg Lie superalgebra $\mathfrak{hei}(0|2N)$ can be embedded into the Clifford superalgebra $C(2N)$ and, correspondingly, it has a representation in the Lie superalgebra $\mathrm{End}(\mathbb{C}^{2^{N-1}|2^{N-1}})$, which is related to the spin representation of $\mathfrak{o}(2N+1, \mathbb{C})$ in $\mathrm{End}(\mathbb{C}^{2^N})$, see [5,21]. This representation allows us to realize the Lie superalgebra $\mathfrak{spo}(2|2N)$, which preserves a non-degenerate super skew-symmetric form on a $(2|2N)$-dimensional superspace, as a subsuperalgebra of $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$. We prove that if $N = 1, 2$ and 3, then $\mathfrak{spo}(2|2N)$ and the loop algebra of $\mathfrak{o}(2N, \mathbb{C})$ generate a subsuperalgebra of $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$, which is isomorphic to $K(2)$, $\hat{K}'(4)$ and $CK_6$, respectively. If $N \geq 4$, then the generated superalgebra is the entire $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$. Using this fact, we prove that if $N \geq 4$, then there is no embedding of $K(2N)$ into $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$.

In conclusion, we would like to point out that embeddings of superconformal algebras into Lie superalgebras of pseudodifferential symbols on a supercircle and into Lie superalgebras of matrices over a Weyl algebra (which are closely related to each other) are only possible for superconformal algebras, which are in a sense, exceptional, and they do not occur in the general case. This singles out exceptional superconformal algebras from all superconformal algebras. It would be interesting to give a rigorous mathematical formulation of this fact.

## 2. Preliminaries

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$, and let $\Lambda(1, 2N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$ be the associative superalgebra with natural multiplication and with the following parity of generators: $p(t) = \bar{0}$, $p(\xi_i) = p(\eta_i) = \bar{1}$ for $i = 1, \ldots, N$. Let $W(2N)$ be the Lie superalgebra of all superderivations of $\Lambda(1, 2N)$. Let $\partial_t$, $\partial_{\xi_i}$ and $\partial_{\eta_i}$ stand for $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \xi_i}$ and $\frac{\partial}{\partial \eta_i}$, respectively. By definition,

$$K(2N) = \{D \in W(2N) \mid D\Omega = f\Omega \text{ for some } f \in \Lambda(1, 2N)\}, \tag{1}$$

where $\Omega = \mathrm{d}t + \sum_{i=1}^{N} \xi_i d\eta_i + \eta_i d\xi_i$ is a differential 1-form, which is called a *contact form*, see [3,4,9,13]. Recall that $K(2N)$ can be described in terms of pseudodifferential symbols on $S^{1|N}$, see [17,18]. Consider the *Poisson superalgebra of pseudodifferential symbols*

$$P(2N) = P \otimes \Lambda(2N), \tag{2}$$

where the Poisson algebra $P$ is formed by the formal series of the form

$$A(t, \tau) = \sum_{i=-\infty}^{k} a_i(t)\tau^i, \tag{3}$$

where $k$ is some integer, $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the even variable $\tau$ corresponds to $\partial_t$, see [14]. The Poisson super bracket is defined as follows:

$$\{A, B\} = \partial_\tau A \partial_t B - \partial_t A \partial_\tau B + (-1)^{p(A)+1} \sum_{i=1}^{N} (\partial_{\xi_i} A \partial_{\eta_i} B + \partial_{\eta_i} A \partial_{\xi_i} B). \tag{4}$$

Note that there exists an embedding

$$K(2N) \subset P(2N), \qquad N \geq 0. \tag{5}$$

Consider a $\mathbb{Z}$-grading of the associative superalgebra

$$P(2N) = \oplus_{i\in\mathbb{Z}} P_{(i)}(2N) \tag{6}$$

defined by $\deg_{\mathrm{Lie}} f = \deg f - 1$, where $\deg f$ is defined by

$$\begin{aligned} &\deg t = \deg \eta_i = 0 \quad \text{for } i = 1, \ldots, N, \\ &\deg \tau = \deg \xi_i = 1 \quad \text{for } i = 1, \ldots, N. \end{aligned} \tag{7}$$

With respect to the Poisson super bracket,

$$\{P_{(i)}(2N), P_{(j)}(2N)\} \subset P_{(i+j)}(2N). \tag{8}$$

Thus $P_{(0)}(2N)$ is a subsuperalgebra of $P(2N)$. We proved in [17] that $P_{(0)}(2N)$ is isomorphic to $K(2N)$. Note that $K(2N)$ is spanned by $2^{2N}$ fields, one of which is a Virasoro field. Recall that a Lie superalgebra is called simple if it contains no nontrivial ideals [7]. If $N \neq 2$, then $K(2N)$ is simple. If $N = 2$, then $K(4)$ is not simple. In this case the derived Lie superalgebra $K'(4) = [K(4), K(4)]$ is an ideal in $K(4)$ of codimension one, defined from the exact sequence

$$0 \to K'(4) \to K(4) \to \mathbb{C}t^{-1}\tau^{-1}\xi_1\xi_2\eta_1\eta_2 \to 0, \tag{9}$$

and $K'(4)$ is simple. Thus $K'(4)$ is spanned by 16 fields inside $P(4)$. Each field consists of elements, which are indexed by $n$, where $n$ runs through $\mathbb{Z}$. These fields are

$$\begin{aligned} &L_n = t^{n+1}\tau, \quad Q_n = t^{n+1}\tau\eta_1\eta_2, \quad X_n^i = t^{n+1}\tau\eta_i, \\ &Y_n^i = t^n\xi_i, \quad R_n^{ji} = t^n\eta_j\xi_i, \quad Z_n^i = t^n\eta_1\eta_2\xi_i, \quad i, j = 1, 2, \end{aligned} \tag{10}$$

$$\begin{aligned} &G_n^0 = t^{n-1}\tau^{-1}\xi_1\xi_2, \quad G_n^i = t^{n-1}\tau^{-1}\xi_1\xi_2\eta_i, \quad i = 1, 2, \\ &G_n^3 = nt^{n-1}\tau^{-1}\xi_1\xi_2\eta_1\eta_2, \quad n \neq 0. \end{aligned} \tag{11}$$

Note that $K'(4)$ has three independent central extensions [13]. $K(6)$ contains the exceptional superconformal algebra $CK_6$ as a subsuperalgebra [3,6,9–12,22–24]. $CK_6$ has no non-trivial central extensions [3]. In [18] we obtained a realization of $CK_6$ in terms of pseudodifferential symbols on $S^{1|3}$, and proved that $CK_6$ is spanned by 32 fields inside $K(6) \subset P(6)$. Each field consists of elements indexed by $n$, where $n$ runs through $\mathbb{Z}$. Explicitly, $CK_6$ is spanned by the following 20 fields:

$$\begin{aligned} &L_n = t^{n+1}\tau, \quad G_n^i = t^{n+1}\tau\eta_i, \quad \text{where } i = 1, 2, 3, \\ &\tilde{G}_n^i = t^n\xi_i - nt^{n-1}\tau^{-1}\eta_j\xi_i\xi_j, \quad \text{where } i = 1, j = 2 \text{ or } i = 2, j = 3 \text{ or } i = 3, j = 1, \\ &T_n^{ij} = t^n\eta_i\xi_j - nt^{n-1}\tau^{-1}\eta_k\eta_i\xi_k\xi_j, \quad \text{where } i, j, k \in \{1, 2, 3\} \text{ and } i \neq j \neq k, \\ &J_n^{ij} = t^{n+1}\tau\eta_i\eta_j, \quad \text{where } 1 \leq i < j \leq 3, \\ &\tilde{J}_n^{ij} = t^{n-1}\tau^{-1}\xi_i\xi_j, \quad \text{where } 1 \leq i < j \leq 3, \\ &I_n = t^{n+1}\tau\eta_1\eta_2\eta_3, \end{aligned} \tag{12}$$

and the following 12 fields, where $i = 1, j = 2, k = 3$ or $i = 2, j = 3, k = 1$ or $i = 3, j = 1, k = 2$:

$$\begin{aligned} &T_n^i = -t^n(\eta_j\xi_j + \eta_k\xi_k) + nt^{n-1}\tau^{-1}\eta_j\eta_k\xi_j\xi_k, \\ &S_n^i = -t^n\eta_i(\eta_j\xi_j + \eta_k\xi_k) + nt^{n-1}\tau^{-1}\eta_i\eta_j\eta_k\xi_j\xi_k, \\ &\tilde{S}_n^i = t^{n-1}\tau^{-1}(\eta_j\xi_j - \eta_k\xi_k)\xi_i, \\ &I_n^i = t^{n-1}\tau^{-1}\eta_i\xi_j\xi_k, \end{aligned}$$

Note that $L_n$ is a Virasoro field.

## 3. Lie superalgebras of matrices over a Weyl algebra

By definition, a *Weyl algebra* is

$$\mathcal{W} = \sum_{i \geq 0} \mathcal{A} d^i, \tag{13}$$

where $\mathcal{A}$ is an associative commutative algebra and $d : \mathcal{A} \to \mathcal{A}$ is a derivation of $\mathcal{A}$, with the relations

$$da = d(a) + ad, \quad a \in \mathcal{A}, \tag{14}$$

See [15,16] for further details. Set

$$\mathcal{A} = \mathbb{C}[t, t^{-1}], \quad d = \partial_t. \tag{15}$$

Let $\mathrm{End}(\mathcal{W}^{m|n})$ be the complex Lie superalgebra of matrices of size $(m+n) \times (m+n)$ over $\mathcal{W}$. Let $\mathfrak{spo}(2|2N)$ be a Lie superalgebra, which preserves an even non-degenerate super skew-symmetric form on the $(2|2N)$-dimensional superspace.

**Lemma 3.1.** *For each $N \geq 1$, there exists an embedding*

$$\mathfrak{spo}(2|2N) \subset \mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}}). \tag{16}$$

**Proof.** Let $V = \mathrm{Span}(\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N)$. Let $\mathfrak{hei}(0|2N)$ be the Heisenberg Lie superalgebra: $\mathfrak{hei}(0|2N)_{\bar{1}} = V$ with the non-degenerate symmetric bilinear form $(\xi_i, \eta_i) = (\eta_i, \xi_i) = 1$, and $\mathfrak{hei}(0|2N)_{\bar{0}} = \mathbb{C}C$, where $C$ is a central element in $\mathfrak{hei}(0|2N)$. Let $C(2N)$ be the *Clifford superalgebra* with generators $\xi_i, \eta_i$ and relations

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = \delta_{i,j} - \xi_j \eta_i, \quad i, j = 1, \ldots, N. \tag{17}$$

Let

$$\iota : \mathfrak{o}(2N, \mathbb{C}) \ni \mathfrak{hei}(0|2N) \to C(2N), \tag{18}$$

where $\mathfrak{o}(2N, \mathbb{C}) \cong \Lambda^2(V)$, be an embedding given by

$$\begin{aligned} &\iota(\xi_i \xi_j) = \xi_i \xi_j, \quad \iota(\eta_i \eta_j) = \eta_i \eta_j, \quad \iota(\xi_i \eta_j) = \xi_i \eta_j, \quad i \neq j, \\ &\iota(\xi_i \eta_i) = \xi_i \eta_i - \frac{1}{2}, \quad \iota(\xi_i) = \xi_i, \quad \iota(\eta_i) = \eta_i, \quad \iota(C) = 1. \end{aligned} \tag{19}$$

Note that $C(2N) \cong \mathrm{End}(\mathbb{C}^{2^{N-1}|2^{N-1}})$. The elements $\xi_i$ act by multiplication on the superspace $\Lambda(\xi_1, \ldots, \xi_N)$, and $\eta_i$ acts as $\partial_{\xi_i}$. Hence there exists an embedding

$$\rho : \mathfrak{o}(2N, \mathbb{C}) \ni \mathfrak{hei}(0|2N) \to \mathrm{End}(\mathbb{C}^{2^{N-1}|2^{N-1}}). \tag{20}$$

Note that if we consider $V$ as an *even* vector space, then formulas (19) define an embedding of $\mathfrak{o}(2N+1, \mathbb{C}) \cong \Lambda^2(V) \oplus V$ into the *Clifford algebra* $C(2N)$, and correspondingly, the spin representation of $\mathfrak{o}(2N+1, \mathbb{C})$ in $\mathrm{End}(\mathbb{C}^{2^N})$, see [5,21].

Let

$$\mathrm{End}(\mathbb{C}^{2^{N-1}|2^{N-1}}) = \mathrm{End}_{-1} \oplus \mathrm{End}_0 \oplus \mathrm{End}_1, \tag{21}$$

where $\mathrm{End}_0$ is the set of even complex matrices, and $\mathrm{End}_{-1}$ and $\mathrm{End}_1$ are the sets of odd upper triangular matrices and odd lower triangular matrices, respectively. Let $X \in V$. Then $\rho(X) = \rho(X)_{-1} + \rho(X)_1$, where $\rho(X)_{\pm 1} \in \mathrm{End}_{\pm 1}$. Define

$$\rho(X)^{\pm} \in \mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}}) \tag{22}$$

by setting

$$\rho(X)^{\pm} = \rho(X)^{\pm}_{-1} + \rho(X)^{\pm}_{1}, \tag{23}$$

where

$$\begin{aligned}\rho(X)^{\pm}_{-1} &= t^{\pm 1}\rho(X)_{-1},\\ \rho(X)^{\pm}_{1} &= \left(tdt^{\pm 1} \mp \frac{1}{2}t^{\pm 1}\right)\rho(X)_{1}.\end{aligned} \tag{24}$$

Define $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ by setting

$$\begin{aligned}\mathfrak{g}_{\bar{1}} &= \rho(V)^{\pm},\\ \mathfrak{g}_{\bar{0}} &= \rho(\mathfrak{o}(2N,\mathbb{C})) \oplus \mathfrak{sl}(2),\end{aligned} \tag{25}$$

where $\mathfrak{sl}(2) = \mathrm{Span}(E, H, F)$, and $E$, $H$ and $F$ are the following diagonal matrices in $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$:

$$\begin{aligned}E &= \frac{1}{2}\mathrm{i}\left(\left(tdt^2 - \frac{1}{2}t^2\right)1_{2^{N-1}}\middle|\left(t^2dt - \frac{1}{2}t^2\right)1_{2^{N-1}}\right),\\ F &= \frac{1}{2}\mathrm{i}\left(\left(tdt^{-2} + \frac{1}{2}t^{-2}\right)1_{2^{N-1}}\middle|\left(dt^{-1} + \frac{1}{2}t^{-2}\right)1_{2^{N-1}}\right),\\ H &= (td)1_{2^{N-1}|2^{N-1}},\end{aligned} \tag{26}$$

so that the standard commutation relations hold:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Then $\mathfrak{g} \cong \mathfrak{spo}(2|2N)$. $\square$

Let $\tilde{\mathfrak{o}}(2N,\mathbb{C}) = \mathrm{Span}(t^n\rho(X) \mid X \in \mathfrak{o}(2N,\mathbb{C}), n \in \mathbb{Z})$. Thus $\tilde{\mathfrak{o}}(2N,\mathbb{C})$ is isomorphic to the loop algebra of $\mathfrak{o}(2N,\mathbb{C})$.

**Theorem 3.2.** *If $N = 1$, then $\mathfrak{spo}(2|2)$ and $\tilde{\mathfrak{o}}(2,\mathbb{C})$ generate $K(2)$,*
*if $N = 2$, then $\mathfrak{spo}(2|4)$ and $\tilde{\mathfrak{o}}(4,\mathbb{C})$ generate $\hat{K}'(4)$,*
*if $N = 3$, then $\mathfrak{spo}(2|6)$ and $\tilde{\mathfrak{o}}(6,\mathbb{C})$ generate $CK_6$,*
*if $N \geq 4$, then $\mathfrak{spo}(2|2N)$ and $\tilde{\mathfrak{o}}(2N,\mathbb{C})$ generate* $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$.

**Proof.** *Case $N = 1$.* It is easy to see that $\mathfrak{spo}(2|2)_{\bar{1}} = \mathrm{Span}(\rho(\xi_1^{\pm}), \rho(\eta_1^{\pm}))$, where

$$\begin{aligned}&\rho(\xi_1^+) = \left(\begin{array}{c|c} 0 & 0 \\ \hline t & 0 \end{array}\right), \quad \rho(\xi_1^-) = \left(\begin{array}{c|c} 0 & 0 \\ \hline t^{-1} & 0 \end{array}\right),\\ &\rho(\eta_1^+) = \left(\begin{array}{c|c} 0 & t^2d + \frac{1}{2}t \\ \hline 0 & 0 \end{array}\right), \quad \rho(\eta_1^-) = \left(\begin{array}{c|c} 0 & d - \frac{1}{2}t^{-1} \\ \hline 0 & 0 \end{array}\right).\end{aligned} \tag{27}$$

Hence $\mathfrak{spo}(2|2)$ and $\tilde{\mathfrak{o}}(2,\mathbb{C}) = \mathrm{Span}\left(\left(\begin{array}{c|c} -t^n & 0 \\ \hline 0 & t^n \end{array}\right)\right)$ generate the following subsuperalgebra of $\mathrm{End}(\mathcal{W}^{1|1})$:

$$\mathfrak{g} = \mathrm{Span}(L_n, H_n, G_n, \tilde{G}_n \mid n \in \mathbb{Z}),$$

where

$$\begin{aligned}&L_n = \left(\begin{array}{c|c} t^{n+1}d + nt^n & 0 \\ \hline 0 & t^{n+1}d \end{array}\right), \quad H_n = \left(\begin{array}{c|c} -t^n & 0 \\ \hline 0 & t^n \end{array}\right),\\ &G_n = \left(\begin{array}{c|c} 0 & t^{n+1}d + \frac{n}{2}t^n \\ \hline 0 & 0 \end{array}\right), \quad \tilde{G}_n = \left(\begin{array}{c|c} 0 & 0 \\ \hline t^n & 0 \end{array}\right).\end{aligned} \tag{28}$$

The isomorphism

$$\sigma : K(2) \subset P(2) \to \mathfrak{g}$$

is given by

$$\begin{aligned} &\sigma(t^{n+1}\tau) = L_n + \frac{n}{2}H_n, \quad \sigma(t^n\xi_1\eta_1) = \frac{1}{2}H_n, \\ &\sigma(t^n\xi_1) = \tilde{G}_n, \quad \sigma(t^{n+1}\tau\eta_1) = G_n. \end{aligned} \tag{29}$$

Note that $L_n$ is a Virasoro field.

In the cases when $N = 2$ and 3, we will use an embedding of $\hat{K}'(4)$ and $CK_6$, respectively, into a deformation of $P(2N)$. Let $P_1(2N) = P_1 \otimes C(2N)$. The associative multiplication in the vector space $P_1 = P$ is determined as follows (see [14]):

$$A(t,\tau) \circ B(t,\tau) = \sum_{n\geq 0} \frac{1}{n!}\partial_\tau^n A(t,\tau)\partial_t^n B(t,\tau). \tag{30}$$

The product of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$, where $A_1, B_1 \in P_1$, and $X, Y \in C(2N)$, is given by

$$AB = (A_1 \circ B_1) \otimes (XY). \tag{31}$$

The Lie super bracket in $P_1(2N)$ is $[A, B] = AB - (-1)^{p(A)p(B)}BA$. $P_1(2N)$ is called the *Lie superalgebra of pseudodifferential symbols on* $S^{1|N}$, see [17,18].

*Case* $N = 2$. We proved in [17] that $\hat{K}'(4)$ is spanned inside $P_1(4)$ by the 12 fields given in (10) and 4 fields

$$\begin{aligned} &G_n^0 = \tau^{-1} \circ t^{n-1}\xi_1\xi_2, \quad G_n^i = \tau^{-1} \circ t^{n-1}\xi_1\xi_2\eta_i, \quad i = 1, 2, \\ &G_n^3 = n\tau^{-1} \circ t^{n-1}\xi_1\xi_2\eta_1\eta_2 + t^n. \end{aligned} \tag{32}$$

Note that $L_n$ is a Virasoro field. The central element in $\hat{K}'(4)$ is $G_0^3 = 1$, and the corresponding 2-cocycle is

$$\begin{aligned} &c(L_n, G_k^3) = -n\delta_{n+k,0}, \\ &c(X_n^i, G_k^j) = (-1)^j\delta_{n+k,0}, \quad 1 \leq i \neq j \leq 2, \\ &c(Q_n, G_k^0) = \delta_{n+k,0}. \end{aligned} \tag{33}$$

Note that this 2-cocycle is different from the Virasoro 2-cocycle. Let $V^\mu = t^\mu\mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$, where $\mu \in \mathbb{C} \setminus \mathbb{Z}$. Define a representation of $\hat{K}'(4)$ in $V^\mu$ according to the formulas (10) and (32). In particular, $\tau^{-1}$ is identified with an antiderivative, and the central element acts by the identity operator. Consider the following basis in $V^\mu$:

$$\begin{aligned} &v_m^0(\mu) = t^{m+\mu}, \quad v_m^i(\mu) = t^{m+\mu}\xi_i, \quad i = 1, 2, \\ &v_m^3(\mu) = \frac{t^{m+\mu}}{m+\mu}\xi_1\xi_2, \quad m \in \mathbb{Z}. \end{aligned} \tag{34}$$

Explicitly, the action of $\hat{K}'(4)$ on $V^\mu$ is given as follows:

$$
\begin{aligned}
&L_n(v^i_m(\mu)) = (m+\mu)v^i_{m+n}(\mu), \quad i=0,1,2,\\
&L_n(v^3_m(\mu)) = (n+m+\mu)v^3_{m+n}(\mu),\\
&X^i_n(v^i_m(\mu)) = (m+\mu)v^0_{m+n}(\mu), \quad i=1,2,\\
&X^1_n(v^3_m(\mu)) = v^2_{m+n}(\mu),\\
&X^2_n(v^3_m(\mu)) = -v^1_{m+n}(\mu),\\
&Q_n(v^3_m(\mu)) = -v^0_{m+n}(\mu),\\
&Y^i_n(v^0_m(\mu)) = v^i_{m+n}(\mu), \quad i=1,2,\\
&Y^1_n(v^2_m(\mu)) = (n+m+\mu)v^3_{m+n}(\mu),\\
&Y^2_n(v^1_m(\mu)) = -(n+m+\mu)v^3_{m+n}(\mu),\\
&R^{ii}_n(v^0_m(\mu)) = v^0_{m+n}(\mu), \quad i=1,2,\\
&R^{ii}_n(v^j_m(\mu)) = v^j_{m+n}(\mu), \quad R^{ij}_n(v^i_m(\mu)) = -v^j_{m+n}(\mu), \quad i\neq j=1,2,\\
&Z^1_n(v^2_m(\mu)) = -v^0_{m+n}(\mu), \quad Z^2_n(v^1_m(\mu)) = v^0_{m+n}(\mu),\\
&G^0_n(v^0_m(\mu)) = v^3_{m+n}(\mu), \quad G^i_n(v^i_m(\mu)) = v^3_{m+n}(\mu), \quad i=1,2,\\
&G^3_n(v^i_m(\mu)) = v^i_{m+n}(\mu), \quad n\neq 0, i=0,1,2,3.
\end{aligned}
\tag{35}
$$

These formulas remain valid for $\mu = 0$. Thus we obtain a representation of $\hat{K}'(4)$ in the superspace $\mathbb{C}[t,t^{-1}] \otimes \Lambda(\xi_1,\xi_2)$ with a basis

$$\{v^0_m, v^3_m; v^1_m, v^2_m\},$$

where

$$v^0_m = t^m, \quad v^3_m = t^m\xi_1\xi_2, \quad v^i_m = t^m\xi_i, \quad i=1,2, \quad m\in\mathbb{Z}.$$

We have

$$
\begin{aligned}
&L_n(v^i_m) = t^{n+1}dv^i_m, \quad i=0,1,2, \quad L_n(v^3_m) = tdt^n v^3_m\\
&X^i_n(v^i_m) = t^{n+1}dv^0_m, \quad i=1,2, \quad X^1_n(v^3_m) = t^n v^2_m,\\
&X^2_n(v^3_m) = -t^n v^1_m, \quad Q_n(v^3_m) = -t^n v^0_m,\\
&Y^i_n(v^0_m) = t^n v^i_m, \quad i=1,2, \quad Y^1_n(v^2_m) = tdt^n v^3_m,\\
&Y^2_n(v^1_m) = -tdt^n v^3_m, \quad R^{ii}_n(v^0_m) = t^n v^0_m, \quad i=1,2,\\
&R^{ii}_n(v^j_m) = t^n v^j_m, \quad R^{ij}_n(v^i_m) = -t^n v^j_m, \quad i\neq j=1,2,\\
&Z^1_n(v^2_m) = -t^n v^0_m, \quad Z^2_n(v^1_m) = t^n v^0_m,\\
&G^0_n(v^0_m) = t^n v^3_m, \quad G^i_n(v^i_m) = t^n v^3_m, \quad i=1,2,\\
&G^3_n(v^i_m) = t^n v^i_m, \quad n\neq 0, i=0,1,2,3.
\end{aligned}
\tag{36}
$$

This gives a realization of $\hat{K}'(4)$ as a subsuperalgebra of $\mathrm{End}(\mathcal{W}^{2|2})$. Note that

$$\mathfrak{spo}(2|4) \subset \hat{K}'(4) \subset \mathrm{End}(\mathcal{W}^{2|2}), \tag{37}$$

where $\mathfrak{spo}(2|4)_{\bar{1}}$ is spanned by the following elements:

$$
\begin{aligned}
&\rho(\xi_1)^\pm = Y^1_{\pm1} \mp \frac{1}{2}G^2_{\pm1}, \quad \rho(\xi_2)^\pm = Y^2_{\pm1} \pm \frac{1}{2}G^1_{\pm1},\\
&\rho(\eta_1)^\pm = X^1_{\pm1} \pm \frac{1}{2}Z^2_{\pm1}, \quad \rho(\eta_2)^\pm = X^2_{\pm1} \mp \frac{1}{2}Z^1_{\pm1}.
\end{aligned}
\tag{38}
$$

$\tilde{\mathfrak{o}}(4, \mathbb{C})$ is generated by $R_n^{12}, R_n^{21}, G_n^0$ and $Q_n$. When these elements act on $\mathfrak{spo}(2|4)_{\bar{1}}$, they generate the 8 fields $X_n^i, Y_n^i, G_n^i$ and $Z_n^i$, where $i = 1, 2$, which span $\hat{K}'(4)_{\bar{1}}$, and hence $\hat{K}'(4)$ is generated by $\mathfrak{spo}(2|4)$ and $\tilde{\mathfrak{o}}(4, \mathbb{C})$.

*Case* $N = 3$. We proved in [18] that $CK_6$ is spanned inside $P_1(6)$ by the 8 fields: $L_n$, $G_n^i$, $I_n$, and $J_n^{ij}$, given in (12), and the following 24 fields, where $n \in \mathbb{Z}$:

First we have the 12 fields:

$$\begin{aligned}
\tilde{G}_n^i &= t^n \xi_i - n\tau^{-1} \circ t^{n-1} \xi_i \xi_j \eta_j, \quad \text{where } i = 1, j = 2 \text{ or } i = 2, j = 3 \text{ or } i = 3, j = 1,\\
T_n^{ij} &= t^n \eta_i \xi_j - n\tau^{-1} \circ t^{n-1} \xi_k \xi_j \eta_k \eta_i, \quad \text{where } i, j, k \in \{1, 2, 3\} \text{ and } i \neq j \neq k,\\
\tilde{J}_n^{ij} &= \tau^{-1} \circ t^{n-1} \xi_i \xi_j, \quad \text{where } 1 \leq i < j \leq 3,
\end{aligned} \tag{39}$$

We also have the following 12 fields, where $i = 1, j = 2, k = 3$ or $i = 2, j = 3, k = 1$ or $i = 3, j = 1, k = 2$:

$$\begin{aligned}
T_n^i &= -t^n(\eta_j \xi_j + \eta_k \xi_k) + n\tau^{-1} \circ t^{n-1} \xi_j \xi_k \eta_j \eta_k + t^n,\\
S_n^i &= -t^n \eta_i (\eta_j \xi_j + \eta_k \xi_k) + n\tau^{-1} \circ t^{n-1} \xi_j \xi_k \eta_i \eta_j \eta_k + t^n \eta_i,\\
\tilde{S}_n^i &= \tau^{-1} \circ t^{n-1} (\xi_j \xi_i \eta_j - \xi_k \xi_i \eta_k),\\
I_n^i &= \tau^{-1} \circ t^{n-1} \xi_j \xi_k \eta_i.
\end{aligned} \tag{40}$$

Let $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2, \xi_3)$, where $\mu \in \mathbb{C} \setminus \mathbb{Z}$. Define a representation of $CK_6$ in $V^\mu$ according to the formulas (12) and (39)–(40). Consider the following basis in $V^\mu$:

$$\begin{aligned}
v_m^i(\mu) &= t^{m+\mu} \xi_i, \quad \hat{v}_m^i(\mu) = \frac{t^{m+\mu}}{m+\mu} \xi_j \xi_k, \quad 1 \leq i \leq 3,\\
v_m^4(\mu) &= t^{m+\mu}, \quad \hat{v}_m^4(\mu) = -\frac{t^{m+\mu}}{m+\mu} \xi_1 \xi_2 \xi_3,
\end{aligned}$$

where $m \in \mathbb{Z}$ and $(i, j, k)$ is the cycle $(1, 2, 3)$ in the formulas for $\hat{v}_m^i(\mu)$. Explicitly, the action of $CK_6$ on $V^\mu$ is given as follows:

$$\begin{aligned}
&L_n(v_m^i(\mu)) = (m+\mu) v_{m+n}^i(\mu), \quad L_n(\hat{v}_m^i(\mu)) = (m+n+\mu) \hat{v}_{m+n}^i(\mu),\\
&G_n^i(v_m^i(\mu)) = (m+\mu) v_{m+n}^4(\mu), \quad G_n^i(\hat{v}_m^4(\mu)) = -(m+n+\mu) \hat{v}_{m+n}^i(\mu),\\
&G_n^i(\hat{v}_m^k(\mu)) = v_{m+n}^j(\mu), \quad G_n^i(\hat{v}_m^j(\mu)) = -v_{m+n}^k(\mu),\\
&\tilde{G}_n^i(v_m^4(\mu)) = v_{m+n}^i(\mu), \quad \tilde{G}_n^i(\hat{v}_m^i(\mu)) = -\hat{v}_{m+n}^4(\mu),\\
&\tilde{G}_n^i(v_m^k(\mu)) = -(m+n+\mu) \hat{v}_{m+n}^j(\mu), \quad \tilde{G}_n^j(v_m^j(\mu)) = (m+\mu) v_{m+n}^k(\mu),\\
&T_n^{ij}(v_m^i(\mu)) = -v_{m+n}^j(\mu), \quad T_n^{ij}(\hat{v}_m^j(\mu)) = \hat{v}_{m+n}^i(\mu),\\
&T_n^i(v_m^i(\mu)) = -v_{m+n}^i(\mu), \quad T_n^i(v_m^4(\mu)) = -v_{m+n}^4(\mu),\\
&T_n^i(\hat{v}_m^i(\mu)) = \hat{v}_{m+n}^i(\mu), \quad T_n^i(\hat{v}_m^4(\mu)) = \hat{v}_{m+n}^4(\mu),\\
&S_n^i(v_m^i(\mu)) = -v_{m+n}^4(\mu), \quad S_n^i(\hat{v}_m^4(\mu)) = -\hat{v}_{m+n}^i(\mu),\\
&\tilde{S}_n^i(v_m^k(\mu)) = -\hat{v}_{m+n}^j(\mu), \quad \tilde{S}_n^i(v_m^j(\mu)) = -\hat{v}_{m+n}^k(\mu),\\
&I_n^i(v_m^i(\mu)) = \hat{v}_{m+n}^i(\mu), \quad I_n(\hat{v}_m^4(\mu)) = v_{m+n}^4(\mu),\\
&J_n^{ij}(\hat{v}_m^k(\mu)) = -v_{m+n}^4(\mu), \quad J_n^{ij}(\hat{v}_m^4(\mu)) = v_{m+n}^k(\mu),\\
&\tilde{J}_n^{ij}(v_m^4(\mu)) = \hat{v}_{m+n}^k(\mu), \quad \tilde{J}_n^{ij}(v_m^k(\mu)) = -\hat{v}_{m+n}^4(\mu),
\end{aligned} \tag{41}$$

where $(i, j, k)$ is the cycle $(1, 2, 3)$. These formulas remain valid for $\mu = 0$. Thus we obtain a representation of $CK_6$ in the superspace $\mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2, \xi_3)$ with a basis

$$\{\hat{v}_m^i, v_m^4; v_m^i, \hat{v}_m^4\}, \quad i = 1, 2, 3,$$

where

$$\hat{v}^i_m = t^m \xi_j \xi_k, \quad v^4 = t^m, \quad v^i_m = t^m \xi_i, \quad \hat{v}^4_m = -t^m \xi_1 \xi_2 \xi_3,$$

$(i, j, k)$ is the cycle (1, 2, 3) in the formulas for $\hat{v}^i_m$, and $m \in \mathbb{Z}$. We have

$$\begin{aligned}
&L_n(v^i_m) = t^{n+1} d v^i_m, \quad L_n(\hat{v}^i_m) = t d t^n \hat{v}^i_m,\\
&G^i_n(v^i_m) = t^{n+1} d v^4_m, \quad G^i_n(\hat{v}^4_m) = -t d t^n \hat{v}^i_m,\\
&G^i_n(\hat{v}^k_m) = t^n v^j_m, \quad G^i_n(\hat{v}^j_m) = -t^n v^k_m,\\
&\tilde{G}^i_n(v^4_m) = t^n v^i_m, \quad \tilde{G}^i_n(\hat{v}^i_m) = -t^n \hat{v}^4_m,\\
&\tilde{G}^i_n(v^k_m) = -t d t^n \hat{v}^j_m, \quad \tilde{G}^j_n(v^j_m) = t^{n+1} d v^k_m,\\
&T^{ij}_n(v^i_m) = -t^n v^j_m, \quad T^{ij}_n(\hat{v}^j_m) = t^n \hat{v}^i_m,\\
&T^i_n(v^i_m) = -t^n v^i_m, \quad T^i_n(v^4_m) = -t^n v^4_m,\\
&T^i_n(\hat{v}^i_m) = t^n \hat{v}^i_m, \quad T^i_n(\hat{v}^4_m) = t^n \hat{v}^4_m,\\
&S^i_n(v^i_m) = -t^n v^4_m, \quad S^i_n(\hat{v}^4_m) = -t^n \hat{v}^i_m,\\
&\tilde{S}^i_n(v^k_m) = -t^n \hat{v}^j_m, \quad \tilde{S}^i_n(v^j_m) = -t^n \hat{v}^k_m,\\
&I^i_n(v^i_m) = t^n \hat{v}^i_m, \quad I_n(\hat{v}^4_m) = t^n v^4_m,\\
&J^{ij}_n(\hat{v}^k_m) = -t^n v^4_m, \quad J^{ij}_n(\hat{v}^4_m) = t^n v^k_m,\\
&\tilde{J}^{ij}_n(v^4_m) = t^n \hat{v}^k_m, \quad \tilde{J}^{ij}_n(v^k_m) = -t^n \hat{v}^4_m,
\end{aligned} \tag{42}$$

where $(i, j, k)$ is the cycle (1, 2, 3). This gives a realization of $CK_6$ as subsuperalgebra of $\mathrm{End}(\mathcal{W}^{4|4})$. Note that

$$\mathfrak{spo}(2|6) \subset CK_6 \subset \mathrm{End}(\mathcal{W}^{4|4}), \tag{43}$$

where $\mathfrak{spo}(2|6)_{\bar{1}}$ is spanned by the following elements:

$$\begin{aligned}
&\rho(\xi_i)^{\pm} = \tilde{G}^i_{\pm 1} \mp \frac{1}{2} \tilde{S}^i_{\pm 1},\\
&\rho(\eta_i)^{\pm} = G^i_{\pm 1} \mp \frac{1}{2} S^i_{\pm 1}, \quad i = 1, 2, 3.
\end{aligned} \tag{44}$$

The Lie algebra $\tilde{\mathfrak{o}}(6, \mathbb{C})$ is generated by $T^{ij}_n$ $(i \neq j)$, and $J^{ij}_n$, $\tilde{J}^{ij}_n$ $i < j$. Clearly, when these elements act on $\mathfrak{spo}(2|6)_{\bar{1}}$, they generate the 12 fields $G^i_n$, $\tilde{G}^i_n$, $S^i_n$, and $\tilde{S}^i_n$, where $i = 1, 2, 3$. They also generate the 4 fields: $I^i_n$, $i = 1, 2, 3$ and $I_n$, due to the commutation relations

$$\begin{aligned}
&[\tilde{J}^{ij}_n, \rho(\eta_k)^+] = -n I^k_{n+1},\\
&[J^{ij}_n, \rho(\eta_k)^+] = -n I_{n+1},
\end{aligned} \tag{45}$$

where $(i, j, k)$ is the cycle (1, 2, 3), and $J^{ij}_n = -J^{ji}_n$, $\tilde{J}^{ij}_n = -\tilde{J}^{ji}_n$ for $i > j$. Thus they generate the 16 fields which span $(CK_6)_{\bar{1}}$, and hence $CK_6$ is generated by $\mathfrak{spo}(2|6)$ and $\tilde{\mathfrak{o}}(6, \mathbb{C})$.

*Case* $N = 4$. Let $S$ be the subset of $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$, generated by $\mathfrak{spo}(2|8)$ and $\tilde{\mathfrak{o}}(8, \mathbb{C})$. Consider the following basis in $\Lambda(\xi_1, \ldots, \xi_4)$:

$$\begin{aligned}
&\Lambda(\xi_1, \ldots, \xi_4)_{\bar{0}} = \{v_0, v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}, \hat{v}_0\},\\
&\Lambda(\xi_1, \ldots, \xi_4)_{\bar{1}} = \{v_1, \ldots, v_4, \hat{v}_1, \ldots, \hat{v}_4\},
\end{aligned} \tag{46}$$

where

$$v_0 = 1, \quad v_{ij} = \xi_i \xi_j, \quad \hat{v}_0 = \xi_1 \xi_2 \xi_3 \xi_4, \quad v_i = \xi_i, \quad \hat{v}_i = \xi_1 \cdots \hat{\xi}_i \cdots \xi_4.$$

Let $E^{i,j}_{\pm 1}$ be an elementary $8 \times 8$ matrix in $\mathrm{End}_{\pm 1}$. Let

$$\begin{aligned}
&E_1 = \{t^n X, (t^n d) X \mid X = E^{i,j}_1, n \in \mathbb{Z}\},\\
&E_{-1} = \{t^n X \mid X = E^{i,j}_{-1}, n \in \mathbb{Z}\}.
\end{aligned} \tag{47}$$

Note that it suffices to show that $E_{\pm 1} \subset S$. Let $E_0^{i,j}$ and $\tilde{E}_0^{i,j} = E_0^{i+8,j+8}$, where $1 \le i, j \le 8$, be elementary $8 \times 8$ matrices in $\mathrm{End}_0$. Let

$$E_0 = \{t^n X \mid X = E_0^{i,j}, \tilde{E}_0^{i,j}, i \neq j, \quad n \in \mathbb{Z}\}. \tag{48}$$

Note that $E_0 \subset S$. In fact,

$$[t^n \rho(\eta_1\eta_2), \rho(\eta_3)^+] = nt^{n+1} E_1^{1,8}, \tag{49}$$

Similarly, we can show that $t^n E_1^{i,j} \in S$, for

$$\begin{aligned} &i = 1, j = 5, 6, 7, 8, \quad i = 2, j = 3, 4, 5, 6,\\ &i = 3, j = 2, 4, 5, 8, \quad i = 4, j = 1, 4, 6, 7,\\ &i = 5, j = 1, 4, 6, 7, \quad i = 6, j = 1, 3, 6, 8,\\ &i = 7, j = 1, 2, 7, 8, \quad i = 8, j = 1, 2, 3, 4. \end{aligned} \tag{50}$$

Note that

$$\begin{aligned} &[\rho(\xi_3)^+, t^n E_1^{1,8}] = t^{n+1}(E_0^{1,2} + \tilde{E}_0^{3,8}),\\ &[t^n(E_0^{1,2} + \tilde{E}_0^{3,8}), E_1^{2,4}] = t^n E_1^{1,4},\\ &[\rho(\xi_2)^+, t^n E_1^{1,4}] = t^{n+1}\tilde{E}_0^{2,4}. \end{aligned} \tag{51}$$

On the other hand,

$$[\rho(\eta_1)^+, t^n E_1^{2,4}] = t^{n+1}(E_0^{2,4} + \tilde{E}_0^{2,4}). \tag{52}$$

Hence, $t^n E_0^{2,4} \in S$ and $t^n \tilde{E}_0^{2,4} \in S$. Similarly, we can show that all the elements of $E_0$ are in $S$. According to (50), for each fixed $j$, where $1 \le j \le 8$, we have that $t^n E_1^{k,j} \in S$ for some $1 \le k \le 8$. Note also that for each fixed $j$, there exists a $k$ such that $(t^n d)E_1^{k,j} \in S$. For example, the supercommutator

$$[t^n E_0^{5,2}, \rho(\xi_2)^+] = -(t^{n+2}d)E_1^{5,1} \tag{53}$$

produces such an element for $j = 1$. Obviously, using supercommutators of $t^n E_0^{i,k}$ with $t^n E_1^{k,j}$ and $(t^n d)E_1^{k,j}$, we obtain that $t^n E_1^{i,j} \in S$ and $(t^n d)E_1^{i,j} \in S$ for any $i$. Hence $E_1 \subset S$. Finally, for each $\xi_i$ and $\eta_i$, we now have that $\rho(\xi_i)^+_{-1} \in S$ and $\rho(\eta_i)^+_{-1} \in S$. From these matrices we can obtain all matrices $t^n E_{-1}^{i,j}$ using supercommutators with $t^n \tilde{E}_0^{i,j}$. Hence $E_{-1} \subset S$. Thus $S$ coincides with $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$.

*Case* $N > 4$. Induction on $N$. Assume that the statement is proved for $N - 1$. Let $C(2N - 2)$ be the Clifford superalgebra with generators $\xi_i$ and $\eta_i$, where $i = 1, \dots, N - 1$. Present $\Lambda(\xi_1, \dots, \xi_N)$ as follows:

$$\begin{aligned} &\Lambda(\xi_1, \dots, \xi_N)_{\bar 0} = \Lambda(\xi_1, \dots, \xi_{N-1})_{\bar 1}\xi_N \oplus \Lambda(\xi_1, \dots, \xi_{N-1})_{\bar 0},\\ &\Lambda(\xi_1, \dots, \xi_N)_{\bar 1} = \Lambda(\xi_1, \dots, \xi_{N-1})_{\bar 1} \oplus \Lambda(\xi_1, \dots, \xi_{N-1})_{\bar 0}\xi_N. \end{aligned} \tag{54}$$

Let

$$\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}}) = \mathrm{End}_{-1} \oplus \mathrm{End}_0 \oplus \mathrm{End}_1. \tag{55}$$

Let $E_{\pm 1}^{i,j}$ $(1 \le i, j \le 2^{N-1})$ be an elementary $2^{N-1} \times 2^{N-1}$ matrix in $\mathrm{End}_{\pm 1}$, and let $E_0^{i,j}$ and $\tilde{E}_0^{ij} = E_0^{i+2^{N-1}, j+2^{N-1}}$ be elementary matrices in $\mathrm{End}_0$. By the inductive hypothesis, $\mathfrak{spo}(2|2N - 2)$ and $\tilde{\mathfrak{o}}(2N - 2, \mathbb{C})$ generate

$$\mathrm{End}(\mathcal{W}^{2^{N-2}|2^{N-2}}) = \mathrm{End}'_{-1} \oplus \mathrm{End}'_0 \oplus \mathrm{End}'_1, \tag{56}$$

where

$$\begin{aligned}
\mathrm{End}'_{-1} &= \mathrm{Span}(\mathcal{W}E_{-1}^{i,j+2^{N-2}}) \subset \mathrm{End}_{-1},\\
\mathrm{End}'_{1} &= \mathrm{Span}(\mathcal{W}E_{1}^{i+2^{N-2},j}) \subset \mathrm{End}_{1},\\
\mathrm{End}'_{0} &= \mathrm{Span}(\mathcal{W}E_{0}^{i+2^{N-2},j+2^{N-2}}, \mathcal{W}\tilde{E}_{0}^{i,j}) \subset \mathrm{End}_{0},
\end{aligned} \tag{57}$$

where $1 \le i, j \le 2^{N-2}$. Then $\mathfrak{o}(2N, \mathbb{C})$ and $\mathrm{End}(\mathcal{W}^{2^{N-2}|2^{N-2}})$ generate $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$. □

Note that we can realize $\mathfrak{spo}(2|2N)$ as a subsuperalgebra of $K(2N) \subset P(2N)$. $\mathfrak{spo}(2|2N)_{\bar{1}}$ is spanned by the elements

$$\begin{aligned}
&(\xi_i)_K^+ := t\xi_i + \frac{1}{2}\tau^{-1}\left(\sum_{j=1}^{N}\eta_j\xi_j\right)\xi_i, \quad (\xi_i)_K^- := t^{-1}\xi_i - \frac{1}{2}t^{-2}\tau^{-1}\left(\sum_{j=1}^{N}\eta_j\xi_j\right)\xi_i,\\
&(\eta_i)_K^+ := t^2\tau\eta_i + \frac{1}{2}t\eta_i\left(\sum_{j=1}^{N}\eta_j\xi_j\right) + \frac{1}{2}\tau^{-1}\eta_i\left(\sum_{j,k=1}^{N}\eta_j\xi_j\eta_k\xi_k\right),\\
&(\eta_i)_K^- := \tau\eta_i - \frac{1}{2}t^{-1}\eta_i\left(\sum_{j=1}^{N}\eta_j\xi_j\right) + \frac{1}{2}t^{-2}\tau^{-1}\eta_i\left(\sum_{j,k=1}^{N}\eta_j\xi_j\eta_k\xi_k\right),
\end{aligned} \tag{58}$$

where $i = 1, \ldots, N$. Let

$$\sigma : \mathfrak{spo}(2|2N) \longrightarrow \mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}}) \tag{59}$$

be an embedding defined by

$$\sigma((\xi_i)_K^{\pm}) = \rho(\xi_i)^{\pm}, \quad \sigma((\eta_i)_K^{\pm}) = \rho(\eta_i)^{\pm}, \quad i = 1, \ldots, N, \quad N \ge 1. \tag{60}$$

Note that embeddings of $K(2)$, $\hat{K}'(4)$ and $CK_6$, obtained in Theorem 3.2, are extensions of $\sigma$, see (29), (38) and (44).

**Corollary 3.3.** *The embedding (59) cannot be extended to an embedding of $K(2N)$ into* $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$ *if $N \ge 4$.*

**Proof.** Suppose that there exists an embedding

$$\sigma : K(2N) \to \mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}}). \tag{61}$$

Then $\sigma(\mathfrak{spo}(2|2N))$ and $\mathrm{Span}(t^n\sigma(\mathfrak{o}(2N, \mathbb{C})))$ must generate a subsuperalgebra of $\sigma(K(2N))$, which is not possible, since they generate the entire $\mathrm{End}(\mathcal{W}^{2^{N-1}|2^{N-1}})$. □

**Remark 3.4.** Note that certain exceptional simple finite-dimensional Lie superalgebras can also be realized as subsuperalgebras of matrices over $\mathcal{W}$. In [19], we obtained a realization of the family $D(2, 1; \alpha)$ as $4 \times 4$ matrices over $\mathcal{W}$. Recall that $F(4)$ is an exceptional finite-dimensional Lie superalgebra such that

$$F(4)_{\bar{0}} = \mathfrak{o}(7) \oplus \mathfrak{sl}(2), \quad F(4)_{\bar{1}} = spin_7 \otimes \mathfrak{sl}(2),$$

see [7] for details. It can be constructed using Clifford algebra techniques, see [21]. We conjecture that $F(4)$ can be realized as a subsuperalgebra of matrices of size $16 \times 16$ over $\mathcal{W}$, so that $F(4)_{\bar{1}}$ is in the span of the matrix fields generated by $\rho(\xi_i)^{\pm}$ and $\rho(\eta_i)^{\pm}$, $i = 1, \ldots, 4$, under the action of $\tilde{\mathfrak{o}}(8, \mathbb{C})$.

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